Stabilitätsanalyse des Runge-Kutta-Zeitintegrationsschemas für das konvektionserlaubende Modell COSMO-DE

DACH-Tagung
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GB FE, Deutscher Wetterdienst, Offenbach
COSMO-DE (LMK)

Basic time integration scheme for dynamical core: 3-stage Runge-Kutta
(Wicker, Skamarock, 2002, MWR)

Efficiency achieved by:
• time-splitting:
  slow process with large time step $\Delta T = 25$ sec.
  fast process with small time step $\Delta t = 4.16$ sec
• Implicit vertical discretization

Questions:
• Which stability statements can be made?
• Stability analysis in time and space
• Temporal order of the integration scheme?
• Possible improvements/alternatives?

non-hydrostatic resolved convection
$\Delta x = 2.8$ km
421 $\times$ 461 $\times$ 50 GP
$\Delta t = 25$ sec., $T = 21$ h
(since 16. April 2007)
The current 3-stage RK-scheme in the COSMO-model

Wicker, Skamarock (2002) MWR

solve the implicit scheme:
\[ \frac{\phi - \phi^n}{\Delta t} = \beta A_z(\phi^*) + (1 - \beta) A_z(\phi^n) + A_x(\phi^n) + P(\phi^n) \]

... and define its tendency:
\[ L(\phi^n) := \frac{\phi^n - \phi^*}{\Delta t} \]

1. RK-substep:
\[ \phi^* = \phi^n + \Delta t L(\phi^n) \]

fast waves with tendency
\[ \frac{\phi^* - \phi^n}{\Delta t/3}, \text{ starting at } \phi^n \Rightarrow \phi^* \]

solve:
\[ \frac{\phi - [\alpha \phi^n + (1 - \alpha)\phi^*]}{\Delta t} = \beta A_z(\phi) + (1 - \beta) A_z(\phi^*) + A_x(\phi^*) + P(\phi^n) \]

... and define its tendency:
\[ L(\phi^n) := \text{lhs. of the above expression} \]

2. RK-substep:
\[ \phi^{**} = \phi^n + \frac{\Delta t}{2} L(\phi^*) \]

fast waves with tendency
\[ \frac{\phi^{**} - \phi^n}{\Delta t/2}, \text{ starting at } \phi^n \Rightarrow \phi^{**} \]

solve:
\[ \frac{\phi - [\alpha \phi^n + (1 - \alpha)\phi^{**}]}{\Delta t} = \beta A_z(\phi) + (1 - \beta) A_z(\phi^{**}) + A_x(\phi^{**}) + P(\phi^n) \]

... and define its tendency:
\[ L(\phi^{**}) := \text{lhs. of the above expression} \]

3. RK-substep:
\[ \phi^{n+1} = \phi^n + \Delta t L(\phi^{**}) \]

fast waves with tendency
\[ \frac{\phi^{n+1} - \phi^n}{\Delta t}, \text{ starting at } \phi^n \Rightarrow \phi^{n+1} \]

In the following: no vertical advection considered $\leftrightarrow \alpha=0$
Temporal order conditions for the time-split RK3

slow processes (e.g. advection):

\[ Q_s = 1 + \Delta T P_s^{(1)} + \frac{\Delta T^2}{2!} P_s^{(2)} + \frac{\Delta T^3}{3!} P_s^{(3)} + O(\Delta T^4) \]

fast processes (e.g. sound, gravity wave expansion):

\[ Q_f = 1 + \Delta t P_f^{(1)} + \frac{\Delta t^2}{2!} P_f^{(2)} + \frac{\Delta t^3}{3!} P_f^{(3)} + O(\Delta t^4) \]

Insertion into the time-split RK3WS integration scheme:

\[ Q_{RK3WS} = 1 + \Delta T \left[ P_s^{(1)} + P_f^{(1)} \right] + \frac{\Delta T^2}{2} \left[ P_s^{(2)} + (P_s^{(1)})^2 + P_s^{(1)} P_f^{(1)} + \right. \]
\[ \left. + \left( 1 - \frac{1}{n_s} \right) P_f^{(1)} P_s^{(1)} + \left( 1 - \frac{1}{n_s} \right) (P_f^{(1)})^2 + \frac{1}{n_s} P_f^{(2)} \right] + O(\Delta T)^3 \]

compare with \( \exp \left[ \Delta T \left( P_s + P_f \right) \right] \rightarrow \) only 2nd order for:
slow process = Euler forward \text{ and } n_s \rightarrow \infty; \) fast process can be of 2nd order
Linear advection equation (1-dim.)

\[
\frac{\partial q}{\partial t} + u \frac{\partial q}{\partial x} = 0 \quad \text{discretized:} \quad \frac{q_j^{n+1} - q_j^n}{\Delta t} = f_j(q^n)
\]

Spatial discretizations of the advection operator (order 1 ... 6) (u>0 assumed)

- \( f_j^{(1)}(q) := -u \frac{q_j - q_{j-1}}{\Delta x} \) upwind 1st order
- \( f_j^{(2)}(q) := -u \frac{q_{j+1} - q_{j-1}}{2 \Delta x} \) centered diff. 2nd order
- \( f_j^{(3)}(q) := -u \frac{2q_{j+1} + 3q_j - 6q_{j-1} + q_{j-2}}{6 \Delta x} \)
- \( f_j^{(4)}(q) := -u \frac{-(q_{j+2} - q_{j-2}) + 8(q_{j+1} - q_{j-1})}{12 \Delta x} \)
- \( f_j^{(5)}(q) := -u \frac{-3q_{j+2} + 30q_{j+1} + 20q_j - 60q_{j-1} + 15q_{j-2} - 2q_{j-3}}{60 \Delta x} \)
- \( f_j^{(6)}(q) := -u \frac{(q_{j+3} - q_{j-3}) - 9(q_{j+2} - q_{j-2}) + 45(q_{j+1} - q_{j-1})}{60 \Delta x} \)

from the Theorem and Lemma below:
all \( N \)th order (LC-) Runge-Kutta-schemes have the same linear stability properties for \( u=\text{const.} \) !

Hundsdorfer et al. (1995) JCP
Wicker, Skamarock (2002) MWR
Stable Courant-numbers for advection schemes

<table>
<thead>
<tr>
<th></th>
<th>up1</th>
<th>cd2</th>
<th>up3</th>
<th>cd4</th>
<th>up5</th>
<th>cd6</th>
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<td>2.261</td>
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Stability limit for the 'effective Courant-number' for advection schemes

\[ C_{\text{eff}} := \frac{C}{s}, \quad s = \text{stage of RK-scheme} \]

<table>
<thead>
<tr>
<th></th>
<th>up1</th>
<th>cd2</th>
<th>up3</th>
<th>cd4</th>
<th>up5</th>
<th>cd6</th>
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<td>0.159</td>
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→ General theory for Linear Case (LC)-Runge-Kutta-schemes, applied to the Dahlquist test problem
### Stability analysis of a 2D (horizontal + vertical) system with Sound + Buoyancy + Advection (+ Smoothing, Filtering)

#### Advektion

\[
\frac{\partial u}{\partial t} + U_0 \frac{\partial u}{\partial x} = -\frac{1}{\rho_0} \frac{\partial p'}{\partial x} + Q_x
\]

\[
\frac{\partial w}{\partial t} + U_0 \frac{\partial w}{\partial x} = -\frac{1}{\rho_0} \frac{\partial p'}{\partial z} + g \left( \frac{T'}{T_0} - \frac{p'}{p_0} \right) + Q_z
\]

\[
\frac{\partial p'}{\partial t} + U_0 \frac{\partial p'}{\partial x} = \frac{c_p}{c_v} p_0 D + \rho_0 g w
\]

\[
\frac{\partial T'}{\partial t} + U_0 \frac{\partial T'}{\partial x} = -\frac{R}{c_v} T_0 D - \frac{\partial T_0}{\partial z} w
\]

\[D = \frac{\partial u}{\partial x} + \frac{\partial w}{\partial z}\]

### Von Neumann stability analysis

- Restrictions:
  - no boundaries (wave expansion in $\infty$ extended medium)
  - base state: $p_0=\text{const}, T_0=\text{const}$ (for the coefficients)
    (but stratification $dT_0/dz \neq 0$ possible)
  - application to structures with a vertical extension $<< 10$ km
  - no orography (i.e. no metric terms)
  - only horizontal advection
Stabilität einzelner Wellen für Advektion + Schall
(partial operator splitting mit RK3, $C_{\text{adv}}=1.0$, $C_{\text{snd},x}=0.6$)

(ohne Div.-dämpfung)

$\beta^s = 0.5$

$\beta^s = 0.7$

Sound term discretization:

Spatial: centered differences

Temporal:
Forward-backward \textit{(Mesinger, 1977)}
Vertically implicit ($\beta^s \geq \frac{1}{2}$)

$\rightarrow$ stable for $C_{\text{snd},x} < 1$, $C_{\text{snd},z}$ arbitrary
Divergence damping

\[ \frac{\partial \mathbf{v}}{\partial t} = \ldots + \alpha_{div} \nabla \text{div} \mathbf{v} \]

leads to a diffusion of divergence \( \rightarrow \) damping of sound waves

Isotropic div. damping recommended: Gassmann, Herzog (2007) MWR

Courant-numbers:

\[ C'_{\text{div},x} := \alpha_{\text{div}} \frac{\Delta t}{\Delta x^2} \quad \quad C'_{\text{div},z} := \alpha_{\text{div}} \frac{\Delta t}{\Delta z^2} \]

2D, explicit, stagg. grid: stable for \( C_{\text{div},x} + C_{\text{div},z} < \frac{1}{2} \)

2D, hor. expl.-vert. impl., stagg. grid: stable for \( C_{\text{div},x} < \frac{1}{2} \), \( C_{\text{div},z} \) arbitrary

\[ C_{\text{div},x} = 0.1 \text{ für COSMO-DE } \rightarrow \alpha_{\text{div}} = 160000 \text{ m}^2/\text{s} \]

Stabilität einzelner Wellen für Advektion + Schall
(partial operator splitting mit RK3, $C_{\text{adv}}=1.0$, $C_{\text{snd,x}}=0.6$)

$\beta_s=0.5$

$\beta_s=0.7$
Buoyancy terms:

\[
\begin{align*}
\frac{w^{n+1} - w^n}{\Delta t} &= g \left( \beta_T^b \frac{T^{n+1}}{T_0} + (1 - \beta_T^b) \frac{T^n}{T_0} - \beta_p^b \frac{p^{n+1}}{p_0} - (1 - \beta_p^b) \frac{p^n}{p_0} \right) \\
\frac{p^{n+1} - p^n}{\Delta t} &= \rho_0 g (\beta_3^b w^{n+1} + (1 - \beta_3^b) w^n) \\
\frac{T^{n+1} - T^n}{\Delta t} &= -\frac{\partial T_0}{\partial z} (\beta_4^b w^{n+1} + (1 - \beta_4^b) w^n)
\end{align*}
\]

acoustic cut-off frequency \( \omega_a := \sqrt{N^2 + \frac{g z^2}{c_s^2}} \); \( C_{\text{buoy}} := \omega_a \Delta t \)

\[
C^b = \frac{1}{T_0} \frac{\partial T_0}{\partial z} \frac{c_s^2}{g} \approx -0.24 \text{ (standard atmosphere)}
\]

<table>
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<tr>
<th>fully explicit</th>
<th>unstable</th>
<th>-</th>
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<tr>
<td>forward-backward</td>
<td>stable for ( C_{\text{buoy}} &lt; 2 )</td>
<td>neutral</td>
</tr>
<tr>
<td>Crank-Nicholson ( \beta = 1/2 )</td>
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<td>neutral</td>
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<tr>
<td>Crank-Nicholson ( \beta &gt; 1/2 )</td>
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<tr>
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</table>
Stabilität einzelner Wellen für Advektion + Schall + Auftrieb
(partial operator splitting mit RK3, $C_{adv}=1.0$, $C_{snd,x}=0.6$)

ohne Div.-dämpfung  
mit Div.-dämpfung

$\beta^s=0.5$

$\beta^s=0.7$
Choice of CN-parameters for buoyancy in the $p'\Delta T$-Dynamic of COSMO-DE

$\beta = 0.5$ (purely Crank-Nicolson)

$\beta = 0.6$

$\beta = 0.7$

$\beta = 0.8$

$\beta = 0.9$

$\beta = 1.0$ (purely implicit)

$\Rightarrow$ choose $\beta = 0.7$ as the best value
'Simplest'-LC-RK4
(4-stage, 2nd order)

\[
\begin{array}{c|cccc}
0 & 1/4 & 1/4 & 0 & 0 & 0 & 1 \\
1/3 & 0 & 1/3 \\
1/2 & 0 & 0 & 1/2 \\
\end{array}
\]

Are 4-stage RK-schemes competitive?
S-LC-RK3
(3-stage, 2nd order) + up5
\[ \Delta T/\Delta t = 6 \]
no smoothing

S-LC-RK4 (4-stage, 2nd order) + cd4
\[ \Delta T/\Delta t = 12 \]
no smoothing
+ 4th order diffusion

instab. by long waves

instab. by short waves

instab. by long waves
**Are 4-stage RK-schemes competitive?**

### 'Simplest'-LC-RK4
(4-stage, 2nd order)

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### 'Classical' RK4
(4-stage, 4th order)

(Numerical recipies)

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### RK-SSP(4,3)
(4-stage, 3rd order)

(Ruuth, Spiteri, 2004)

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</tbody>
</table>
Other  RK4 + cd4 + add. smoothing ...

**S-LC-RK4**
(4-stage, 2nd order)

**SSP(4,3)-RK**
4-stage, 3rd order
(Ruuth, Spiteri, 2004)
is not a LC-RK-scheme!

**'classical' RK4**
(4-stage, 4th order)

Strong stability preserving schemes do not automatically work together with timesplitting

Completely unstable (!?) due to negative coefficients in the 'substepping-form'?
Summary

• RK3 time-splitting: only of 2nd order in time for Euler-forward advection and $n_s \to \infty$; fast processes (sound, gravity waves) in 2nd order helps for finite $n_s$

• von Neumann stability-analysis of 2D Euler equations (Sound-Buoyancy-Advection) in time and space:
  • without metric terms: no off-centering for sound needed ($\beta_s = \frac{1}{2}$)
    (with metric: some off-centering recommendable)
  • $C_{div} \sim 0.1$ is recommended
  • buoyancy needs off-centering ($\beta_B = 0.7$, and even then is slightly unstable)

• RK4 + cd4 + weak add. smoothing could be a more efficient alternative to RK3 + up5; with better advection properties than RK3 + up3

Baldauf, 2010: Linear stability analysis of Runge-Kutta based partial time-splitting schemes for the Euler equations, MWR