A General Analytical Expression for the Radiation Source Function of Emitting and Scattering Media within the Matrix Operator Method

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Abstract
We derived an analytical expression for the radiation source function for a thermally emitting and scattering medium within the Matrix-Operator-Method (MOM). The final formulation is equivalent to the one found by Aronson and Yarmush (1966), who applied the transfer matrix to gamma-ray and neutron penetration (Aronson and Yarmush, 1966) and to transport problems in slab geometry (Aronson, 1970). For the thermal infrared case, the general analytical expression reduces to a simple formula, which depends only on the zenith angle. The formula is incorporated in the MOM together with analytical expressions of the transmission and reflection operators following Liu (1990). With the aid of these formulations, expressions are derived as parameterizations of the scattering effects of clouds in non-scattering radiative transfer models by a modification of the emissivity and transmittance of clouds. The accuracy is better than 0.5% in the 11.5 μm window region for clouds of arbitrary optical depths.

1 Introduction

The matrix operator method is one of the most commonly used methods for solving radiative transfer problems (cf. e.g. Plass et al., 1973; Fischer and Grassl, 1984). With this method, the scattering problems of real atmospheres are solved by discretizing the atmosphere into several homogeneous layers, for each of which transmission and reflection operators are calculated in general by the adding or doubling method (cf. e.g. Stephens, 1976; Plass et al., 1973). The final solution is then obtained by the star product method (Redheffer, 1962). As to longwave infrared radiation, the calculation of the source function for the thermally emitting and scattering medium is very time-consuming. An elegant and elaborate mathematical solution has been developed for transport problems with sources by Aronson and Yarmush (1966) and Aronson (1970). In this paper, a different derivation is outlined (Appendix A) and a general analytical expression of the radiation source function within MOM is provided. The application requires analytical formulations for the transmission and reflection operators. We use the expressions found by Liu (1990), which relate directly to the phase matrix rather than to eigenvalues and eigenvectors of the phase matrix as do the formulations of Aronson (1970). The analytical expressions of the transmission and the reflection operators, and the source function are shown to have the correct limits for optical depth $\tau = 0$ and for single scattering albedos $\omega = 0$ and $\omega = 1$. 
With the analytical expressions and for 16 x 16 discrete zenith angles (this is equivalent to a 16 stream method) the computer time is about one eighth as compared to the adding or doubling method. By incorporating the expressions into the MOM the overall computation time is largely reduced. For operational processing of satellite data, however, the necessary computations are still too time-consuming. With the commonly used limb darkening approximation and the derived analytical expressions, the effects of the scattering and reflection by clouds can be easily accounted for even in non-scattering radiation models. In the following, the expressions derived assume a Fourier expansion of the radiation field with respect to the azimuth angle.

2 An Analytical Expression for the Source Function of the Atmosphere

For an emitting and non-scattering atmospheric layer, the radiative transfer equation can be written in the form (Liou, 1980):

$$\mu = \frac{d \Gamma(\tau, \mu)}{d \tau} = I(\tau, \mu) - B(T)$$ (1)

with I, the radiance, or the radiation intensity, \(\mu = \cos(\theta)\), \(\theta\) the zenith angle, \(B(T)\), the Planck function, \(T\), the constant temperature of the layer, and \(\tau\), the optical depth.

The solution of Eq. (1) for one homogeneous atmospheric layer of total optical depth \(\tau\) is

$$\Gamma(0, -\mu) = \exp(-\tau/\mu) \Gamma(-\tau, -\mu) +$$

$$+ \left[1 - \exp(-\tau/\mu)\right] B(T) =$$

$$= T(\tau, \mu) \Gamma(-\tau, -\mu) + J^-(-\tau, -\mu)$$ (3)

$$\Gamma(0, -\mu) = \exp(-\tau/\mu) \Gamma(-\tau, -\mu) +$$

$$+ \left[1 - \exp(-\tau/\mu)\right] B(T) =$$

$$= T(\tau, \mu) \Gamma(-\tau, -\mu) + J^-(-\tau, -\mu)$$ (3)

where \(T(\tau, \mu) = \exp(-\tau/\mu)\), the path transmittance of the layer, and

$$J^-(\tau, \mu) = J^-(-\tau, -\mu) = \left[1 - \exp(-\tau/\mu)\right] B(T)$$

the source function of the layer.

The superscripts + and — denote upward and downward directions, respectively. The source function is the contribution from the atmospheric layer itself to the upward and downward radiation intensity at its top and bottom, respectively.

If scattering is included, the radiative transfer through the layer (see Figure 1) satisfies the more complex matrix formulation

$$\begin{bmatrix} \Gamma^+(\tau) \\ \Gamma^+(0) \end{bmatrix} = \begin{bmatrix} T & R \\ R & T \end{bmatrix} \begin{bmatrix} \Gamma^-(\tau) \\ \Gamma^-(0) \end{bmatrix} + \begin{bmatrix} J^+ \\ J^- \end{bmatrix}$$ (4)

with

\(T\): transmission matrix, \(R\): reflection matrix, \(I^\pm\): intensity matrices, \(J^\pm\): source function matrices.

\(T, R, I^\pm\) and \(J^\pm\) have a dimension of \(n \times n\) with \(n\) as the number of zenith angles within one hemisphere. Eq. (4) differs from Eqs. (2) and (3) in the following three points:

1. Integration over zenith angles is involved.
2. An additional term \(R\) is included due to scattering.
3. The source function matrices \(J^\pm\) include scattering effects.

Aronson (1970) has given a general analytical expression for the source function. We derive a formulation for the thermal radiation of emitting and scattering media within MOM, in a way more familiar to the atmospheric science community. The source function matrix \(J\) can be written as (see Appendix A)

$$J = J^+ + J^- = \left[ E - T - R \right] B(T) \mathbf{1}$$ (5)

where \(\mathbf{1} = (1, 1, \ldots, 1)\), and \(\mathbf{1}\) is a vector, all components of which are 1.

The analytical expressions of the \(T\) and \(R\), as given by Liu (1990), are

$$T = 2 \left[ \cos h( \mathbf{H} \tau ) - \mathbf{V} \sin h( \mathbf{H} \tau ) + \cos h( \mathbf{F} \tau ) - \mathbf{U} \sin h( \mathbf{F} \tau ) \right]^{-1}$$ (6)

$$R = \frac{1}{2} \left[ \cos h( \mathbf{H} \tau ) + \mathbf{V} \sin h( \mathbf{H} \tau ) - \cos h( \mathbf{F} \tau ) - \mathbf{U} \sin h( \mathbf{F} \tau ) \right] \mathbf{T}$$ (7)

with

$$\mathbf{F}^2 = (\alpha_1 + \alpha_2) (\alpha_1 - \alpha_2), \quad \mathbf{H}^2 = (\alpha_1 - \alpha_2) (\alpha_1 + \alpha_2)$$

$$\mathbf{V} = (\alpha_1 + \alpha_2), \quad \mathbf{H}^{-1}, \quad \mathbf{U} = (\alpha_1 - \alpha_2) \mathbf{F}^{-1}$$

$$\alpha_1 = \omega \mathbf{M}^{-1} \mathbf{P}^{++} \mathbf{C} - \mathbf{M}^{-1}$$ (9)

$$\alpha_2 = \omega \mathbf{M}^{-1} \mathbf{P}^{--} \mathbf{C}$$ (10)
and ω, the single scattering albedo,

\[ \mathbf{P}^{++} = \{P^{++}(μ_i, μ_j)\}, \]

forward scattering phase function

\[ \mathbf{P}^{+-} = \{P^{+-}(μ_i, -μ_j)\}, \]

backward scattering phase function

\[ \mathbf{M} = [μ_1, δ i j], \]

quadrature points

\[ \mathbf{C} = [C_1, δ i j], \]

integration weights

\[ \mathbf{E}, \]

the n x n unity-matrix.

All columns of the source function matrix \( \mathbf{J} \) equal each other, which means that each element of \( \mathbf{J} \) depends only on the observing zenith angle. The column of \( \mathbf{J} \) or source function vector \( \mathbf{S} \) can be written as

\[
\mathbf{S} = \begin{bmatrix}
\mathbf{s}(θ_1) \\
\mathbf{s}(θ_2) \\
\vdots \\
\mathbf{s}(θ_n)
\end{bmatrix} = \begin{bmatrix}
1 \\
1 \\
\vdots \\
1
\end{bmatrix} \mathbf{B}(T)
\]

with \( \mathbf{B}(T) \) being the emissivity of the medium being given by

\[
\mathbf{B}(T) = \begin{bmatrix}
\varepsilon(θ_1) \\
\varepsilon(θ_2) \\
\vdots \\
\varepsilon(θ_n)
\end{bmatrix}
\]

(11)

The layer is assumed to be horizontally homogeneous and in local thermodynamic equilibrium. In the thermal infrared region, atmospheric layers are the only radiation sources, which depend only on the observing zenith angle. Thus the radiance \( I \) depends also only on the observing zenith angle. Algorithms using the matrix operator method to calculate longwave radiation have been contributed by many authors (e.g. Stephens, 1976; Grant and Hunt, 1969a, 1969b; Stamnes and Swanson, 1981; Waterman, 1981). To review some of the striking properties of cirrus clouds in the longwave radiation region we compare results from water and ice clouds using the derived analytical expressions. We select a tropical summer, a dry stratosphere, a standard atmospheric

\[ \omega = 1 \) and purely absorbing (i.e. \( \omega = 0 \) ) media. For \( T = 0 \), \( \mathbf{T} \) should reduce to the unity matrix and \( \mathbf{R} \) must vanish. This is obvious by noticing

\[
\begin{align*}
\sin h(\mathbf{H}τ)|_{τ=0} &= \sin h(\mathbf{F}τ)|_{τ=0} = 0 \\
\cos h(\mathbf{H}τ)|_{τ=0} &= \cos h(\mathbf{F}τ)|_{τ=0} = \mathbf{E}
\end{align*}
\]

For purely absorbing media (\( \omega = 0 \)), the constant matrices in Eqs. (6) and (7) reduce to

\[ \mathbf{H} = \mathbf{F} = \mathbf{M}^{-1} \]

\[ \mathbf{U} = \mathbf{V} = -\mathbf{E} \]

For this case, Eqs. (6) and (7) give

\[ \mathbf{T} = \exp(-\mathbf{M}^{-1}τ), \quad \mathbf{R} = \mathbf{0} \]

For purely scattering media (\( \omega = 1 \)), we should have

\[
\begin{bmatrix}
1 \\
1 \\
\vdots \\
1
\end{bmatrix} = \begin{bmatrix}
1 \\
1 \\
\vdots \\
1
\end{bmatrix}
\]

(12)

because energy must be conserved. This point can be easily proved having analytical formulations for \( \mathbf{R} \) and \( \mathbf{T} \). A proof is provided in Appendix C.

We compared the analytical formulations of \( \mathbf{R} \) and \( \mathbf{T} \) with the computations using the doubling or adding methods for a large range of \( τ \) and \( \omega \). The results agree to better than 0.01 %. The differences can be attributed to the limited accuracy of the computer and the numerical scheme.

### 4 Scattering Effects in the Thermal Infrared

In the range of the thermal infrared radiation, the effects of scattering on the radiation field in water clouds are small and can be neglected in most cases. This is different for ice clouds. The effects of reflection and scattering of high cirrus clouds on the outgoing radiation can be up to 20 % (Platt and Stephens, 1980). Since the source function of the atmosphere in the thermal radiation region depends only on the zenith angle, the radiative intensities also depend only on the zenith angle. Algorithms using the matrix operator method to calculate longwave radiation have been contributed by many authors (e.g. Stephens, 1976; Grant and Hunt, 1969a, 1969b; Stamnes and Swanson, 1981; Waterman, 1981). To review some of the striking properties of cirrus clouds in the longwave radiation region we compare results from water and ice clouds using the derived analytical expressions. We select a tropical summer, a dry stratosphere, a standard atmospheric...
profile with one homogeneous ice cloud layer (183–247 hPa) and one homogeneous water cloud layer (500–630 hPa), respectively. The surface albedo is set to zero to facilitate the interpretation. The full characterization of cloud microphysical properties and the determination of the optical properties associated with these microphysical parameters is impossible, even with the most sophisticated experimental design (Stone et al., 1990). To further simplify the problem, we assume that the clouds are composed of spherical particles distributed according to the stratocumulus cloud model of Hansen (1971)

\[ n(r) = N_0 A r^{5.474} \exp(-1.5899 r) \]  

with \( n \), number of particles in units of \( \text{cm}^{-3} \mu\text{m}^{-1} \), \( A = 0.07318 \mu\text{m}^{-1} \), and \( N_0 \), the total number concentration per \( \text{cm}^3 \). Although \( r \) is the particle radius in \( \mu\text{m} \), the function \( r^{5.474} \exp(-1.5899 r) \) is treated as a dimensionless function. The refractive index of ice and water was taken from Irvine and Pollack (1968) and from Hale and Querry (1973), respectively. The radiation transfer model used for the non-scattering case was a narrowband model developed by Schmitz (1986). The description of the gas absorption within his model is also used in applying the Matrix-Operator-Method. The water-vapour line transmittance is treated by an exponential-sum fit (Wiscombe and Evans, 1977) and the water-vapour continuum absorption is taken from Grassl (1976). The following computations are performed with a single scattering albedo \( \omega = 0.937 \) at 11.5 \( \mu\text{m} \).

The calculations with the Matrix-Operator-Method show that the reflectance of a thick ice cloud layer can be larger than 20 % and increases with the viewing zenith angle (Figure 2). For large optical depths the reflection approaches a non-zero constant and the transmission goes to zero (Figure 3). Figure 4 shows the variation of the brightness temperature at the top of the atmosphere with the number concentration of the particles for scattering and non-scattering clouds. The scattering effect reduces the brightness temperature up to 32 K (at \( \tau = 10 \)) and approaches 15 K for large optical depths. For an opaque cloud, there is no penetration of energy from below the cloud. The upward intensity at the top of the atmosphere equals \( \varepsilon(\theta) B(T_c) \) (\( T_c \): the temperature of the cloud layer) when contributions from the atmospheric layer above the cloud are neglected (see Eq. (11)). \( \varepsilon(\theta) \) equals 1 without scattering and is less than 1 with scattering. Due to scattering, the upward energy below the cloud is partly reflected by the cloud. Therefore, the scattering and the reflection reduce the upward fluxes above the opaque cloud and increase the downward fluxes below the cloud (see Figure 5). The deduced cloud temperature from satellite observations is lower than the actual temperature of the cloud when scattering is neglected. In this sense, the cloud height assignment of ice clouds will be overestimated by a radiation transfer model that does not

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**Figure 2** Variation of transmittance (dashed line) and reflectance (solid line) with zenith angle \( \Theta \) for an ice cloud layer of spherical particles at 11.5 \( \mu\text{m} \). The single scattering albedo \( \omega \) is 0.937. The optical depth \( \tau \) is 5.

**Figure 3** Variation of transmittance (dashed line) and the reflectance (solid line) with the optical depth \( \tau \) for an ice cloud layer of spherical particles at 11.5 \( \mu\text{m} \). The single scattering albedo \( \omega \) is 0.937 and zenith angle \( \Theta = 61 \text{ degree} \).
account for the reflection of the clouds, which is the common situation in the present operational methods using satellite observations. It is expensive, however, to directly use the scattering radiation model in the operational scheme. Therefore, a simple radiation model with the inclusion of the effects of scattering and reflection is required and will be given in the next section.

5 The Parameterization of the Reflection and Transmission Matrices in Non-Scattering Models

For the infrared window, the upward radiation at the top of the atmosphere with one ice cloud layer is mainly composed of the upward radiation from the cloud layer (see Eq. (11)) and the upward radiation (coming from below the cloud and traversing it). The effects of the atmosphere above the cloud are generally small and can be neglected. The same is true for scattering effects of the atmosphere below the cloud. Thus we can write the following approximate equation

\[ I^* (\theta_i) + \sum_{j=1}^{n} T_{ij} (\theta_i, \theta_j) I_b (\theta_j) = B(T_c) I_b (\theta_i) + \sum_{j=1}^{n} T_{ij} (\theta_i, \theta_j) I_b (\theta_j) \]

with \( I_b \), the upward radiation below the cloud, and \( B(T_c) \) the Planck function for the cloud temperature \( T_c \). Applying the limb darkening approximation

\[ I_b (\theta_i) = a_1 + a_2 (1/\cos (\theta_i) - 1) I_b (0) \]

each component of the column \( I^* \) in Eq. (14) can be rewritten as

\[ I^* (\theta_i) = \epsilon (\theta_i) B(T_c) + I_b (\theta_i) \sum_{j=1}^{n} \frac{T_{ij} (\theta_i, \theta_j)}{a_1 + a_2 (1/\cos (\theta_i) - 1)} \]

\[ = \epsilon (\theta_i) B(T_c) + \sum_{j=1}^{n} \frac{T_{ij} (\theta_i, \theta_j)}{a_1 + a_2 (1/\cos (\theta_i) - 1)} I_b (\theta_j) \]

with the effective transmittance

\[ T_e (\theta_i, \delta_c) = \left[ a_1 + a_2 (1/\cos (\theta_i) - 1) \right] \sum_{j=1}^{n} \frac{T_{ij} (\theta_i, \theta_j)}{a_1 + a_2 (1/\cos (\theta_j) - 1)} \]

\[ \sum_{j=1}^{n} \frac{T_{ij} (\theta_i, \theta_j)}{a_1 + a_2 (1/\cos (\theta_j) - 1)} \]

\[ \sum_{j=1}^{n} \frac{T_{ij} (\theta_i, \theta_j)}{a_1 + a_2 (1/\cos (\theta_j) - 1)} \]
Table 1 The look-up table of the effective transmittance (1), the emissivity (2), and the function $\exp (-\tau/\mu)$ (3) for ice cloud of spheric particles at 11.5 $\mu$m. The single scattering albedo $\omega = 0.937$. Where $\mu = \cos(c)$, $\tau$: the optical depth.

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<th>$\mu$</th>
<th>1.000</th>
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<th>0.899</th>
<th>0.792</th>
<th>0.652</th>
<th>0.486</th>
<th>0.299</th>
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The coefficients $a_1$ and $a_2$ can be determined through a regression technique based on exact radiation transfer model calculations. For 11.5 $\mu$m we obtain $a_1 = 1.0$ and $a_2 = 0.0072$. The emissivity $\epsilon$ and effective transmittance $T_e$ depend only on the optical depth and the viewing zenith angle for given microphysical properties of the cloud. They can be precalculated for different clouds in form of look-up tables. Table 1 lists the emissivity, the effective transmittance and the transmittance without scattering for the ice cloud described above.

We can include the effect of the atmosphere above the cloud to first order when we write Eq. (16) in the form

$$I^+ (\theta_i) = T_a T_{ij} (\theta_i, \tau_a) \epsilon (\theta_i) B (T_c) + T_c (\theta_i, \tau_c) T_e (\theta_i, \tau_e) I_b (\theta_i)$$

with $T_a$, the atmospheric transmittance of the optical depth $\tau_a$ above the cloud. Eq. (18) has the same form and needs almost the same computation time as a conventional non-scattering radiation model. Figure 6 shows the deviations between using the matrix opera-
General analytical expression for the radiation source function

1.0
0.6
0.2
-0.2
-0.6
-1.0
0 20 40 60 80 100
Optical Depth \( \tau \)

**Figure 6** Variation of deviations of the brightness temperature at the top of the atmosphere for zenith angle \( \Theta = 61^\circ \) with the optical depth \( \tau \) for an ice cloud layer of spherical particles at wavelength \( \lambda = 11.5 \mu m \). The deviations are the differences of the results from the matrix operator method and the parameterized non-scattering radiation model (see Eq. (18)).

200
400
600
800
1000

0 200 400 600
Pressure [hPa]

200 400 600

0 200 400 600
Longwave Fluxes [W/m^2]

**Figure 7** Vertical distribution of the longwave fluxes in an atmosphere for the tropical summer, a dry stratosphere, and standard atmospheric profiles with a water cloud layer of spherical particles. The cloud extends from 500 to 630 hPa. The cloud number concentration is 500 cm\(^{-3}\). The dashed line represents the results from the radiation model calculations including scattering. The solid line represents the results from the radiation model calculations without scattering.

300 400 500

0 0.2 0.4 0.6 0.8
Reflectance

10 100

0 0.2 0.4 0.6 0.8
Wavelength \( \lambda \) [\( \mu m \)]

**Figure 8** Variation of reflectances of an ice cloud layer with wavelength \( \lambda \) (unit: \( \mu m \)) for particle number concentration \( N_0 \) = 1000 cm\(^{-3}\) at zenith angle \( \Theta = 61^\circ \). The dashed line corresponds to the drop size distribution as given by Eq. (19). The solid line corresponds to the drop size distribution according to Eq. (13).

6 Conclusion

Utilizing the analytical expressions for transmission, reflection and the source function, the computation time with the matrix operator method is largely reduced. The expressions give the correct limits and allow further analytical treatments. The calculation show that the reflection effects of ice clouds, with the drop size distribution given by Eq. (13), can be larger than 20%. The reflection effects strongly depend on both the size distribution of cloud particles and the wavelength. For the following size distribution (Deirmendjian, 1969), which is more typical of high cirrus clouds,

\[
n(r) = N_0 A r^6 \exp(-1.89736 \tau r) \quad (19)
\]

with \( A = 6.29 \times 10^7 \, \text{cm}^{-1} \) and mode radius of 40 \( \mu m \), the reflection effect is less than 5% in the infrared window (Figure 8). Larger reflection effects appear at the larger wavelength because of larger single scattering albedos (Figure 9). By parameterizing the source method and Eq. (18). The deviation is less than 0.7 K. Thus the parameterization in form of Eq. (18) is accurate enough for operational processing of the meteorological satellite data for ice clouds. For water droplet clouds, the effects of scattering and reflection can be neglected (Figure 7).
Figure 9 Variation of single scattering albedos of an ice cloud layer with wavelength \( \lambda \) (unit \( \mu \text{m} \)) for particle number concentration \( N_0 = 1000 \text{ cm}^{-3} \). The dashed line corresponds to the drop size distribution as given by Eq. (19). The solid line corresponds to the drop size distribution represented by Eq. (13).

Acknowledgements

One of the authors (Q. Liu) has performed this work under funding support granted by the European Space Agency.

Appendix A

Eq. (4) can be rearranged as

\[
\begin{bmatrix}
E - R & I^*(\tau) \\
0 & T
\end{bmatrix}
\begin{bmatrix}
I^*(\tau) \\
I^*(\tau)
\end{bmatrix}
\begin{bmatrix}
J^* \\
J^*
\end{bmatrix}
\]

(A1)

with \( 0 \) the \( n \times n \) null-matrix.

Premultiplying Eq. (A1) by \( \begin{bmatrix} E - R & 0 \\ 0 & T \end{bmatrix}^{-1} \) yields to

\[
\begin{bmatrix}
I^*(\tau) \\
I^*(\tau)
\end{bmatrix}
= \begin{bmatrix}
T - RT^{-1} R T^{-1} & I^*(0) \\
- T^{-1} R T & I^*(0)
\end{bmatrix}
+ \begin{bmatrix}
E - R & J^* \\
0 & - J^*
\end{bmatrix}
\]

(A2)

For the computation of the inverse matrix see Kraus (1987, page 87). Another formulation describing the radiative transfer through an emitting and scattering medium is (Stephens, 1976)

\[
M \frac{d I^*(\tau)}{d \tau} = I^*(\tau) - \omega \left[ P^{++} C I^*(\tau) + P^{--} C I^*(\tau) \right] -
\]

\[
(1 - \omega) B(T) \tilde{1}
\]

(A3)

\[
M \frac{d I^*(\tau)}{d \tau} = I^*(\tau) - \omega \left[ P^{++} C I^*(\tau) + P^{--} C I^*(\tau) \right] -
\]

\[
(1 - \omega) B(T) \tilde{1}
\]

(A4)

with \( \tilde{1} = (1, 1, ..., 1) \), and \( \tilde{1}^i = (1, 1, ..., 1) \).

Eqs. (A3) and (A4) can be combined to

\[
\frac{d}{d \tau} \begin{bmatrix}
I^*(\tau) \\
I^*(\tau)
\end{bmatrix} = - \begin{bmatrix}
- \alpha_1 & \alpha_2 \\
- \alpha_2 & - \alpha_1
\end{bmatrix} \begin{bmatrix}
I^*(\tau) \\
I^*(\tau)
\end{bmatrix} -
\]

\[
(1 - \omega) \begin{bmatrix}
B(T) M^{-1} \tilde{1} \\
B(T) M^{-1} \tilde{1}
\end{bmatrix}
\]

(A5)

Eq. (A5) is a standard form of the matrix differential equation (see Eq. (8.22) of Kraus, 1987). It has the solution

\[
\begin{bmatrix}
I^*(\tau) \\
I^*(\tau)
\end{bmatrix} = \exp \left[ - \begin{bmatrix}
\alpha_1 & \alpha_2 \\
\alpha_2 & \alpha_1
\end{bmatrix} \tau \right] \begin{bmatrix}
I^*(0) \\
I^*(0)
\end{bmatrix} +
\]

\[
\exp \left[ - \begin{bmatrix}
\alpha_1 & \alpha_2 \\
\alpha_2 & \alpha_1
\end{bmatrix} \tau \right] \left[ E_{2n} \right]^{-1} (1 - \omega) \begin{bmatrix}
B(T) M^{-1} \tilde{1} \\
B(T) M^{-1} \tilde{1}
\end{bmatrix}
\]

(A6)

with \( E_{2n} \) the \( 2n \times 2n \) unity-matrix.

The comparison between the first term of the right hand side of Eqs. (A2) and (A6) gives

\[
\begin{bmatrix}
T - RT^{-1} R T^{-1} & I^*(0) \\
- T^{-1} R & I^*(0)
\end{bmatrix} = \exp \left[ - \begin{bmatrix}
\alpha_1 & \alpha_2 \\
\alpha_2 & \alpha_1
\end{bmatrix} \tau \right]
\]

(A7)

Comparing the second term of the right hand side of Eq. (A2) with Eq. (A6) and using Eq. (A7) results in
the identity
\[
\begin{bmatrix}
J' = & E - R \\
- J' = & 0
\end{bmatrix}
\begin{bmatrix}
T - R T^{-1} R T^{-1} \\
- T^{-1} R T^{-1}
\end{bmatrix}
\begin{bmatrix}
E - R
\end{bmatrix}
\begin{bmatrix}
E_2 n
\end{bmatrix}
\]
\[
\begin{bmatrix}
\alpha_1 & \alpha_2 \\
- \alpha_2 & - \alpha_1
\end{bmatrix}
(1 - \omega)
\begin{bmatrix}
B(T) M^{-1} I \\
- B(T) M^{-1} I
\end{bmatrix}
\begin{bmatrix}
\alpha_1 & \alpha_2
\end{bmatrix}
\]
\[
\begin{bmatrix}
T - E R \\
- R E - T
\end{bmatrix}
(1 - \omega)
\begin{bmatrix}
B(T) M^{-1} I \\
- B(T) M^{-1} I
\end{bmatrix}
\begin{bmatrix}
\alpha_1 & \alpha_2
\end{bmatrix}
\]

By using
\[
(1 - \omega) [E - \omega (P^* + P^*) C]^{-1} I = 1
\]
we get
\[
J = J' = J =
- (E - T - R) (\alpha_1 + \alpha_2) M^{-1} (1 - \omega) B (T) I =
- (E - T - R) [E - \omega (P^* + P^*) C]^{-1}
\cdot (1 - \omega) B (T) I =
- (E - T - R) B (T) I
\]
A proof of the Eq. (A9) is provided in the appendix B.

**Appendix B**

We assume
\[
(1 - \omega) [E - \omega (P^* + P^*) C]^{-1} I = 1
\]
Multiplying Eq. (B1) by the matrix
\[
[E - \omega (P^* + P^*) C],
\]
we get
\[
(1 - \omega)
\begin{bmatrix}
1 \\
1 \\
1
\end{bmatrix}
= [E - \omega (P^* + P^*) C]
\begin{bmatrix}
1 \\
1 \\
1
\end{bmatrix}
\]
Since the phase function obeys the normalization condition
\[
\sum_{j=1}^{n} (P_{ij}^* + P_{ij}) C_j = 1
\]
Eq. (B2) is fulfilled and Eq. (B1) is thus proved.

**Appendix C**

If Eq. (7) is used to give
\[
(T + R) = \left[ E + \frac{1}{2} \left( \cos h (H \tau) + V \sin h (H \tau) \right) - \cos h (F \tau) - U \sin h (F \tau) \right] T
\]
We assume
\[
(1, 1, ..., 1) [M C] (T + R) [M C]^{-1} = (1, 1, ..., 1)
\]
That is
\[
(1, 1, ..., 1) [M C]
\]
\[
\begin{bmatrix}
E + \frac{1}{2} [\cos h (H \tau) + V \sin h (H \tau) - \\
- \cos h (F \tau) - U \sin h (F \tau)]
\end{bmatrix}
= (1, 1, ..., 1) [M C] T^{-1} = (1, 1, ..., 1) [M C]
\cdot \frac{1}{2} [\cos h (H \tau) - V \sin h (H \tau) + \\
+ \cos h (F \tau) - U \sin h (F \tau)]
\]
Eq. (C3) can be further simplified to
\[
(1, 1, ..., 1) [M C] \left[ E + \frac{1}{2} [V \sin h (H \tau) + \cos h (F \tau)] \right]
= (1, 1, ..., 1) [M C] \frac{1}{2} [- V \sin h (H \tau) + \cos h (F \tau)]
\]
By using the normalization condition (see (B3) in Appendix B), we get
\[
(1, 1, ..., 1) [M C] (\alpha_1 + \alpha_2) =
= (1, 1, ..., 1) (1 - \omega) C = (0, 0, ..., 0)
\]
for \(\omega = 1\). Then,
\[
(1, 1, ..., 1) [M C] V = (1, 1, ..., 1) [M C] (\alpha_1 + \alpha_2) H^{-1}
= (0, 0, ..., 0)
\]
and
\[
(1, 1, ..., 1) [M C] F^2 = (1, 1, ..., 1) [M C] (\alpha_1 + \alpha_2)
(\alpha_1 - \alpha_2) = (0, 0, ..., 0)
\]
Therefore,
\[
(1, 1, ..., 1) [M C] V \sin h (H \tau) = (0, 0, ..., 0)
\]
\[
(1, 1, ..., 1) [M C] \cos h (F \tau) = (1, 1, ..., 1) [M C]
\]
The right side of the Eq. (C2) equals the left side of the Eq. (C2).
References


